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Orthogonal Harmonic Polynomials on $U(2)$

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Electric charges and free electromagnetic waves are supposed to be described locally with the same wave differential equation. It is only the topology that is considered to be different. The calculated nonlocal $U(2)$ individuals are characterized by polynomials that belong neither to the classical nor to the Szegö polynomials. The construction of the polynomial solution in component form, their orthogonality over singular measures, the relationships to the Jacobi polynomials, Rodriguez formulas, product decomposition, asymptotic formulas, and completeness are presented in some detail. The possibility is discussed of whether this highly nonlocal model for electric charges can have a physical significance. This work is intended to be a first step for the realization of an old idea of Einstein's (and also commented on by Dirac) to start with the electric charge, not with the Planck constant, as the primary concept for quantum theory.

1. INTRODUCTION

This work is based on the following physical motive. Electrodynamics is characterized by two phenomena: transverse waves and point charges. Free waves are described by the homogeneous Maxwell equations,

$$dF = 0, \qquad d^*F = 0 \tag{1}$$

in terms of external differential forms. The 2-form F corresponds to the antisymmetric Maxwell field tensor in more conventional terms.

The charges are considered as sources of the fields and are introduced "by hand" in the right-hand side of one of these equations,

$$dF = 0, \qquad d^*F = \varepsilon\ {}^*j \tag{2}$$

where ${}^*j$ is the current density and ε is a unit system factor ($\varepsilon = 4\pi$ in the Gaussian system).

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If the charge carriers are described by complex fields, then the coupling between charges and field is constructed more intricately (gauge fields). The electromagnetic sector of the Lagrangian L is symbolically denoted as

$$L = -\bar{\psi}\gamma\hbar D\psi - m_0 c\bar{\psi}\psi - (4\varepsilon c)^{-1}FF \tag{3}$$

Besides the presumed affine connection (Minkowski space M_4 with coordinates x), from the requirement of a local gauge invariance $U(1)$,

$$\psi \to e^{i\varphi(x)}\psi, \qquad A \to A - d\varphi \tag{4}$$

one obtains "automatically" a gauge connection that can be characterized by the covariant derivation

$$D\psi = d\psi - (ie/\hbar c)A\psi, \qquad D\psi \to e^{i\varphi(x)}\, D\psi \tag{5}$$

The coupling is realized by the potential 1-form A, with

$$F = dA \tag{6}$$

The electrical charge obtained from global $U(1)$ gauge invariance is a Noether charge.

The phenomenological character of such electrodynamics, put together in such a way, can be seen from the fact that some fundamental questions are left absolutely open, for instance:

(i) There is no hint for a calculation of the universal low-energy coupling constant $\alpha = (e^2/\hbar c)(\varepsilon/4\pi) \approx 1/137$.

(ii) There is no "internal" reason for the existence of several lepton generations [e, μ, τ, ... (?)]; all of them are thought to be pure electromagnetic (or, in modern terms, electroweak) in the same manner.

Thus at present there is no comprehensive internal connection between charges and waves. This fact is in principal not changed by quantization, as is well known.

In this situation the following question seems apt. Can a reasonable electromagnetic theory be constructed only on the basis of the homogeneous equations (1), when individual compact spaces for the leptons and other massive particles are admitted?

This means we are trying to connect waves and charges by the common requirement that both are locally governed by the same differential equations. It is only the global topology that is different for charges and waves.

Remark 1.1. By using fields on individual spaces the particles get a structure described by variables. These must be hidden in a quantum theory, because the latter must not be changed, according to the correspondence principle. According to Bell's hidden variable theorem, this structure must not be local. Therefore, in contradiction to the Kaluza–Klein models or to ideas that try to answer the question of where the internal or charge spaces of gauge field theories should be located, we have to imagine that our compact spaces are existent in (see Section 9) the Minkowski space, each of them conformallly extended to infinity. The internal spaces are "interpenetrating" each other and are thought to be some kind of "boundary conditions" or "flavor" for particles other than leptons. They are, so to speak, *complements* to the points that are thought to be the particles. This concept is supported by the real existence of Coulomb fields which have exactly such properties, and by the principal nonlocal character of quantum mechanics.

The selection of a proper compact space in the framework of electromagnetism is fixed by the following statement. $U(2) \sim S^1 \times S^3$ is, up to overlappings, the only four-dimensional compact Lie group space that permits a maximally two-parameter set of biinvariant metrics ($\sim$ means diffeomorph).

As is well known, the group space $U(2)$ has the following properties: dimension $d = 4$, Eulerian characteristic $\chi = 0$ of course, cohomology group $\mathbf{H}^2 \sim 0$, Poincaré group $\pi_2 \sim 0$.

It follows that all the properties of importance for electrodynamics can be taken in $U(2)$. That is, transversality of waves ($d = 4$), pseudo-Riemannian metrics ($\chi = 0$, Steenrod's theorem, Cartesian product $S^1 \times S^3$), constant scaling between spacelike (S^3) and timelike (S^1) parts, which is realized in M_4 by the constant light velocity c (two parametric set of metrics), global introduction of a potential A according to equation (6) ($\mathbf{H}^2 = 0$), and the opportunity to find fields without singularities (Lie group space); besides, one cannot lose one's way here ($\pi_2 = 0$).

The model "equation (1) on $U(2)$" also seems interesting per se because of its attractive symmetry, in which highly symmetric (S^1, S^3) and antisymmetric (F) elements are connected in a low dimension without singularities.

* * *

The space $U(2)$ is parametrized by biharmonic coordinates (Barut and Raczka, 1977)

$$S^1: \quad x^0 = \tau; \qquad S^3: \quad x^1 = \varphi_1, \quad x^2 = \varphi_2, \quad x^3 = \vartheta_3 \tag{7}$$

with

$$0 \le \tau, \varphi_1, \varphi_2 < 2\pi, \qquad 0 \le \vartheta_3 < \pi/2 \tag{8}$$

The pseudo-Riemannian biinvariant (standard) metric is then

$$ds^2 = a^2\, d\tau^2 - \sin^2 \vartheta_3\, d\varphi_1^2 - \cos^2 \vartheta_3\, d\varphi_2^2 - d\vartheta_3^2 \tag{9a}$$

where the τ-scaling factor a characterizes the relative "sizes" of S^1 and S^3. For the time being we put for simplicity

$$a = 1 \tag{9b}$$

The consequences of equation (9b) cannot be estimated at the present stage. Besides, equation (9a) is not a solution of the Maxwell-Einstein equations.

Using equation (6) for the potential $A = (\phi, A_1, A_2, A_3) = (\phi, \mathbf{A})$, one can write equation (1) as

$$\cot \vartheta_3 \frac{\partial}{\partial \varphi_1}\left(\frac{\partial \phi}{\partial \varphi_1} - \frac{\partial A_1}{\partial \tau}\right) + \tan \vartheta_3 \frac{\partial}{\partial \varphi_2}\left(\frac{\partial \phi}{\partial \varphi_2} - \frac{\partial A_2}{\partial \tau}\right) + \frac{\partial}{\partial \vartheta_3}\left[b\left(\frac{\partial \phi}{\partial \vartheta_3} - \frac{\partial A_3}{\partial \tau}\right)\right] = 0 \tag{10a}$$

$$-\cot \vartheta_3 \frac{\partial}{\partial \tau}\left(\frac{\partial \phi}{\partial \varphi_1} - \frac{\partial A_1}{\partial \tau}\right) + \frac{1}{b}\frac{\partial}{\partial \varphi_2}\left(\frac{\partial A_2}{\partial \varphi_1} - \frac{\partial A_1}{\partial \varphi_2}\right) + \frac{\partial}{\partial \vartheta_3}\left[\cot \vartheta_3\left(\frac{\partial A_3}{\partial \varphi_1} - \frac{\partial A_1}{\partial \vartheta_3}\right)\right] = 0 \tag{10b}$$

$$-\tan \vartheta_3 \frac{\partial}{\partial \tau}\left(\frac{\partial \phi}{\partial \varphi_2} - \frac{\partial A^2}{\partial \tau}\right) + \frac{1}{b}\frac{\partial}{\partial \varphi_1}\left(\frac{\partial A_1}{\partial \varphi_2} - \frac{\partial A_2}{\partial \varphi_1}\right) + \frac{\partial}{\partial \vartheta_3}\left[\tan \vartheta_3\left(\frac{\partial A_3}{\partial \varphi_2} - \frac{\partial A_2}{\partial \vartheta_3}\right)\right] = 0 \tag{10c}$$

$$-b\frac{\partial}{\partial \tau}\left(\frac{\partial \phi}{\partial \vartheta_3} - \frac{\partial A_3}{\partial \tau}\right) + \cot \vartheta_3 \frac{\partial}{\partial \varphi_1}\left(\frac{\partial A_1}{\partial \vartheta_3} - \frac{\partial A_3}{\partial \varphi_1}\right) + \tan \vartheta_3 \frac{\partial}{\partial \varphi_2}\left(\frac{\partial A_2}{\partial \vartheta_3} - \frac{\partial A_3}{\partial \varphi_2}\right) = 0 \tag{10d}$$

with $b = \sin \vartheta_3 \cos \vartheta_3 = \sqrt{-g}$, where g stands for the metric tensor determinant.

We are interested in special wavelike solutions. Using the wave gauge for A,

$$\phi = 0, \qquad \operatorname{div}(S^3)\mathbf{A} = 0 \tag{11}$$

we look for solutions in the "component form"

$$(A_1, 0, 0), \qquad (0, A_2, 0), \qquad (0, 0, A_3) \tag{12}$$

Since equations (10) are linear and homogeneous, the potential (A_1, A_2, A_3) with A_1, A_2, A_3 from Equation (12) is also a solution.

From equations (10b) and (10c) we obtain then for the components (12)

$$\frac{\partial^2 A_1}{\partial \tau^2} - \frac{1}{\cos^2 \vartheta_3}\frac{\partial^2 A_1}{\partial \varphi_2^2} - \frac{1}{\cot \vartheta_3}\frac{\partial}{\partial \vartheta_3}\left(\cot \vartheta_3 \frac{\partial A_1}{\partial \vartheta_3}\right) = 0 \tag{13a}$$

$$\frac{\partial^2 A_2}{\partial \tau^2} - \frac{1}{\sin^2 \vartheta_3}\frac{\partial^2 A_2}{\partial \varphi_1^2} - \frac{1}{\tan \vartheta_3}\frac{\partial}{\partial \vartheta_3}\left(\tan \vartheta_3 \frac{\partial A_2}{\partial \vartheta_3}\right) = 0 \tag{13b}$$

and

$$\partial^2 A_1/\partial \varphi_1\,\partial \tau = 0, \qquad \partial^2 A_1/\partial \varphi_1\,\partial \varphi_2 = 0, \qquad \partial^2 A_1/\partial \varphi_1\,\partial \vartheta_3 = 0$$

$$\partial^2 A_2/\partial \varphi_2\,\partial \tau = 0, \qquad \partial^2 A_2/\partial \varphi_2\,\partial \varphi_1 = 0, \qquad \partial^2 A_2/\partial \varphi_2\,\partial \vartheta_3 = 0$$

The latter also follow from the second of equations (11), that is, $\partial A_1/\partial\varphi_1 = 0$ and $\partial A_2/\partial\varphi_2 = 0$, respectively.

Similarly, for A_3 we obtain

$$b\frac{\partial^2 A_3}{\partial\tau^2} - \cot\vartheta_3 \frac{\partial^2 A_3}{\partial\varphi_1^2} - \tan\vartheta_3 \frac{\partial^2 A_3}{\partial\varphi_2^2} = 0 \tag{13c}$$

$$\frac{\partial}{\partial\vartheta_3}\left(b\frac{\partial A_3}{\partial\tau}\right) = 0, \qquad \frac{\partial}{\partial\vartheta_3}\left(\cot\vartheta_3 \frac{\partial A_3}{\partial\varphi_1}\right) = 0, \qquad \frac{\partial}{\partial\vartheta_3}\left(\tan\vartheta_3 \frac{\partial A_3}{\partial\varphi_2}\right) = 0$$

Provided A_3 is also a function of ϑ_3, it follows that, apart from rather pathological possibilities such as

$$A_3 = \varphi_1 \tan\vartheta_3 + \varphi_2 \cot\vartheta_3 + h(\vartheta_3)$$

A_3 does not depend on φ_1 and φ_2. That leaves $\partial^2 A_3/\partial\tau^2 = 0$, and there is obviously no possibility to get a wavelike solution for A_3 in this way.

Therefore, our compact waves are also transverse. Periodic wave solutions for A_1 and A_2 are obtained with a separation ansatz from equations (13a) and (13b),

$$A_1^{mk} = \exp(i\omega\tau - im\varphi_2)\tilde{y}_{mk}(x), \qquad x =: \cos\vartheta_3 \tag{14a}$$

$$A_2^{mk} = \exp(i\omega\tau - im\varphi_1)\tilde{y}_{mk}(\xi), \qquad \xi =: \sin\vartheta_3 \tag{14b}$$

where m and ω are integers when periodicity is required and $a = 1$.

The existence of such component form solutions is thought to be (i) a general property of electrodynamics and (ii) a special advantage of the parametrization by biharmonic coordinates where the φ_1 and φ_2 lines are geodesics on S^3 for $\varphi_3 = \pi/2$ and $\varphi_3 = 0$, resp.

The y_{mk} are the orthogonal polynomials of interest in the present paper. The index k is an integer that follows from the series breakdown condition as

$$\omega = m + 2k \tag{15}$$

(see below). Substituting

$$z = x^2 = \cos^2\vartheta_3, \qquad \zeta = \xi^2 = \sin^2\vartheta_3 \qquad (z + \zeta = 1) \tag{16}$$

we find that the functions $y_{mk}(z)$ are determined by the following ordinary differential equation of the Sturm-Liouville type:

$$\frac{d}{dz}\left(z\frac{dy}{dz}\right) + \frac{\omega^2}{4}\frac{1}{1-z}y - \frac{m^2}{4}\frac{1}{z(1-z)}y = 0 \tag{17}$$

Comparing with the general Sturm-Liouville equation,

$$[p(z)y']' - q(z)y + (\lambda/4)\rho(z)y = 0 \tag{18}$$

we see that there are two possibilities for the eigenvalues λ and the corresponding weights $\rho(z)$,

$$q_1(z)=\frac{m^2}{4}\frac{1}{z(1-z)}, \qquad \lambda=\omega^2, \qquad \rho_1(z)=(1-z)^{-1} \tag{19a}$$

$$q_2(z)=\frac{-\omega^2}{4}\frac{1}{1-z}, \qquad \lambda=-m^2, \quad \rho_2(z)=z^{-1}(1-z)^{-1} \tag{19b}$$

Both weights are singular in the closed interval [0, 1], so that the polynomials for $m=0$ are neither classical nor Szegö (Szegö, 1959) polynomials. Nevertheless, we obtain in this work, by means of a traditional treatment of equation (17), many properties for them that are typical for classical polynomials, especially for Legendre polynomials. This corresponds to a conjecture by Bateman and Erdélyi (1953).

2. CONSTRUCTION OF THE POLYNOMIAL SOLUTION

We are interested in polynomial solutions of equation (17) in the x-form, i.e., in the polynomials $\tilde{y}(x)$ from the differential equation

$$(1-x^2)x^2\tilde{y}''+(1-x^2)x\tilde{y}'-m^2\tilde{y}+\omega^2x^2\tilde{y}=0 \tag{20}$$

Note that $m=\text{integer}\Leftrightarrow$periodicity in the compact coordinate $\varphi_j\Leftrightarrow$ polynomial requirement for finite $\tilde{y}(x)$.

From the series ansatz

$$\tilde{y}=\sum_{\nu=0}^{\infty} a_\nu x^{\nu+\rho} \tag{21}$$

we obtain

$$\sum_{\nu=0}^{\infty}[(\nu+\rho)^2-m^2]a_\nu x^{\nu+\rho}-\sum_{\nu=0}^{\infty}[(\nu+\rho)^2-\omega^2]a_\nu x^{\nu+\rho+2}=0$$

Comparing the coefficients at x^ρ yields $(\rho^2-m^2)a_0=0$. Therefore, $\rho=m$ (≥ 0 for finite $\tilde{y}$) or $a_0=0$. The latter case gives, from a comparison of the $x^{\rho+1}$ coefficients, $[(\rho+1)^2-m^2]a_1=0$ and so $\rho=m-1$. If we choose $\rho=m$, then we obtain $a_1=0$. Both ways are equivalent. We pursue the first variant: $\rho=m$, $a_1=0$, a_0 available. Continue comparing the coefficients. Then

$$a_{2\kappa}=\frac{(2\kappa-2+m)^2-\omega^2}{(2\kappa+m)^2-m^2}a_{2\kappa-2} \tag{22}$$

and $a_{2\kappa+1}=0$ for $\kappa=1,2,\ldots$. Polynomials are obtained only for the breakdown condition

$$(2\kappa-2+m)^2-\omega^2=0$$

(and m an integer), which leads to the highest index

$$2\kappa=-m+\omega=:2k \tag{23}$$

with nonvanishing coefficients [cf. equation (15)]. [When polynomial-like solutions of equation (20)—cf. also equation (32)—are obtained for nonintegers $m>0$ and $\omega=2k+m$, then k must be an integer.] It follows that

$$a_{2\kappa}=\frac{(\kappa-1-k)(\kappa-1-k+m)}{\kappa(\kappa+m)}a_{2\kappa-2}, \qquad \kappa=1,2,\ldots,k$$

and finally

$$a_{2\kappa}=(-1)^{\kappa}\binom{k}{\kappa}\binom{k+m+\kappa-1}{k-1}a_0\Big/\binom{k+m-1}{k-1}$$

For

$$a_0=\binom{k+m-1}{k-1}$$

we obtain the polynomials quoted in Donth (1984),

$$\tilde{y}_{mk}(x)=x^m\sum_{\kappa=0}^{k}(-1)^{\kappa}\binom{k}{\kappa}\binom{k+m+\kappa-1}{k-1}x^{2\kappa} \tag{24}$$

for integer $m\geq 0$. Using the substitution (16), we can obtain some other forms:

$$y_{mk}(z)=z^{m/2}\sum_{\kappa=0}^{k}(-1)^{\kappa}\binom{k}{\kappa}\binom{k+m+\kappa-1}{k-1}z^{\kappa} \tag{25a}$$

and for $k\geq 1$

$$y_{mk}(z)=z^{m/2}\sum_{\kappa=0}^{k-1}(-1)^{\kappa}\binom{k}{\kappa}\binom{2k+m-\kappa-1}{m+k}(1-z)^{k-\kappa} \tag{25b}$$

$$y_{mk}(z)=z^{m/2}\sum_{\kappa=0}^{k}(-1)^{\kappa}\binom{k}{\kappa}\binom{k+m-1}{m+\kappa}z^{\kappa}(1-z)^{k-\kappa} \tag{25c}$$

Some polynomials with low indices in the form of equation (24) are given in Donth (1984), and in the form of equation (25c) in Donth (1986). For

$k \geq 1$, the y's can be factorized with $(1-z)$ or $(1-x^2)$, respectively, which yields in low order of k

$$
\begin{aligned}
\tilde{y}_{m0}(x) &= x^m \\
\tilde{y}_{m1}(x) &= x^m(1-x^2) \\
\tilde{y}_{02}(x) &= (1-x^2)(1-3x^2) \\
\tilde{y}_{12}(x) &= x(1-x^2)(2-4x^2) \\
\tilde{y}_{22}(x) &= x^2(1-x^2)(3-5x^2) \\
\tilde{y}_{32}(x) &= x^3(1-x^2)(4-6x^2) \\
\tilde{y}_{03}(x) &= (1-x^2)(1-8x^2+10x^4) \\
\tilde{y}_{13}(x) &= x(1-x^2)(3-15x^2+15x^4) \\
\tilde{y}_{23}(x) &= x^2(1-x^2)(6-24x^2+21x^4) \\
\tilde{y}_{04}(x) &= (1-x^2)(1-15x^2+45x^4-36x^6)
\end{aligned}
\tag{26}
$$

Remark 2.1. It may be of interest, with respect to the construction of an electromagnetic vacuum (see Section 9), that the binomial coefficients in the mixed form (25c), say, allow a rather simple statistical interpretation (Donth, 1986):

$\binom{k}{\kappa}$ = number of κ-sets of a k-set (division of k distinguishable boxes in two kinds $\{z, \zeta\}$, κ pieces of z and $k-\kappa$ pieces of ζ)

$\binom{k+m-1}{m+\kappa}$ = number of different distributions (combination with repetition) of $m+\kappa$ nondistinguishable balls in the $k-\kappa$ boxes labeled with ζ

There is some hope that the large binomial coefficients for $k \to \infty$ can be "renormalized" by a rather small coupling constant for the measurable quantities being finite. For $m=0$ the following counting problem has a surprisingly simple solution (Biess *et al.*, 1986): Given k and κ, find the number $A_i(\kappa, \rho)$, $\rho = k - \kappa$, of boxes containing just i balls. The answer is

$$A_i = \rho \binom{\rho+\kappa-i-2}{\kappa-2}, \qquad \rho \geq 2 \tag{27}$$

3. ORTHOGONALITY

Theorem 1. (i) For $k, k' \neq 0$, $k \neq k'$, and any m (not necessarily an integer), y_{mk} and $y_{mk'}$ are orthogonal on $z = [0, 1]$ with the weight $\rho_1 = (1-z)^{-1}$. (ii) For $k, k' \neq 0$, $m \neq m'$ but $\omega = \omega'$, y_{mk} and $y_{m'k'}$ are orthogonal on $z = [0, 1]$ with the weight $\rho_2 = z^{-1}(1-z)^{-1}$.

Proof. Multiply equation (17) for m', k' by $y_{mk}(z)$. Subtract the equation with m', k' and m, k exchanged,

$$\frac{d}{dz}\left(zy_{mk}\frac{dy_{m'k'}}{dz}-zy_{m'k'}\frac{dy_{mk}}{dz}\right)+\frac{(m'+2k')^2-(m+2k)^2}{4(1-z)}y_{m'k'}y_{mk}-\frac{m'^2-m^2}{4z(1-z)}y_{m'k'}y_{mk}=0 \tag{28}$$

Since $y_{mk}(1)=0$ for $k\geq 1$, integration of (28) over [0, 1] gives

$$(\omega'^2-\omega^2)\int_0^1 y_{m'k'}y_{mk}\frac{dz}{1-z}-(m'^2-m^2)\int_0^1 y_{m'y'}y_{mk}\frac{dz}{z(1-z)}=0$$

where $\omega=m+2k$. This implies property (i) for $m'=m$ and (ii) for $\omega'=\omega$.

Remark 3.1. The weight functions ρ_1 and ρ_2 are not summable. There are no moments.

Remark 3.2. As a rule, the functions $y_{m0}=z^{m/2}$ (i.e., $k=0$) do not belong to the orthogonal system (i) or (ii), because of the supposition $k\geq 1$.

4. RELATIONSHIP TO THE JACOBI POLYNOMIALS

Since equation (17) is a differential equation of the Fuchs type (the three singularities at $z=0, 1, \infty$ are regular singular points), and because the y's can be factorized by $(1-z)$, we put

$$y_{mk}(z)=z^{m/2}(1-z)u_{mk}(z) \tag{29}$$

$u_{mk}(z)$ are polynomials of degree $(k-1)$ and are solutions of the following hypergeometric differential equation:

$$z(z-1)u''+[(m+3)z-(m+1)]u'+(m+k+1)(1-k)u=0 \tag{30}$$

The holomorphic solution of this equation (in the neighborhood of $z=0$) can be represented by the hypergeometric series

$$u=F(m+k+1, 1-k, m+1, z) \tag{31}$$

The series breaks down for integer $k\geq 1$. That is, $u=u_{mk}$, which corresponds therefore to a Jacobi polynomial $R_{k-1}^{(m,1)}(z)$ for the interval [0, 1]. Thus, after comparison with equation (24),

$$y_{mk}(z)=-\binom{m+k-1}{k-1}z^{m/2}(1-z)F(m+k+1, 1-k, m+1, z) \tag{32}$$

and we have proved the following:

Theorem 2. $y_{mk}(z)$, $k \geq 1$, has $k-1$ zeros in the open interval $(0, 1)$.

Remark 4.1. An overview of the distribution of the zeros can be obtained from a comparison with the oscillator equation [Sturm's theorem; cf. Arnol'd (1984)].

Let $t = \ln z$. Then equation (17) reads

$$d^2y/dt^2 + \omega_{\text{eff}}^2(t)y = 0$$

where

$$\omega_{\text{eff}}^2 = \frac{\omega^2}{4}\frac{e^t}{1-e^t} - \frac{m^2}{4}\frac{1}{1-e^t}$$

For zeros z_0 and z_0' lying close to one another (as for large k), their distance $\Delta t = \ln(z_0/z_0')$ can be estimated from

$$\Delta t \approx \pi/\omega_{\text{eff}}$$

which yields, near $z = 1$, the following zero distribution:

$$\Delta t \approx \Delta z \approx \pi[(1-z)/k(k+m)]^{1/2}$$

The largest zero is at $z \approx 1 - \pi^2/2k(k+m)$.

For $m \gg k$, all the zeros are located in the narrow interval $(1-4k/m, 1)$ near $z = 1$. This follows from equation (17) written as

$$z\frac{d}{dz}\left(z\frac{dy}{dz}\right) - \frac{m^2}{4}y + (mk+k^2)\frac{zy}{1-z} = 0$$

The first two terms give $y = z^{m/2}$, which can be disturbed by the third term only in this interval. For small z the zero distribution can be discussed in the asymptotic representation (48) [see below].

5. RODRIGUEZ FORMULA AND GENERATING FUNCTION FOR $m = 0$

Theorem 3. There is a Rodriguez formula for the polynomials y_{0k},

$$y_{0k}(z) = \frac{1-z}{k!}\frac{d^k}{dz^k}[(1-z)^{k-1}z^k] \tag{33}$$

Proof. This follows from Theorem 1(i) after a long but elementary calculation.

Remark 5.1. Formula (33) has the same form as for the classical polynomials, although the weight ρ_1 is not summable. Another formula of this type can be obtained from the Jacobi polynomials,

$$y_{mk} = c_{mk} z^{-m/2} \frac{d^{k-1}}{dz^{k-1}} [z^{m+k-1}(z-1)^k] \tag{34}$$

where c_{mk} are constants. These "associated functions" $y_{mk}(z)$, $m > 0$, can also be obtained by integration. Let $v_{mk} = y_{mk}/z^{m/2}$, $k \geq 1$. Then

$$v_{m+1,k} = v_{mk} + \frac{k-1}{z^{m+1}} \int_0^z dz'\, z'^m v_{mk}(z') \tag{35}$$

Theorem 4. (Generating function). The functions $y_{0k}(z)$, $k = 0, 1, 2, \ldots$, are the coefficients of a Taylor series expansion of

$$H(z, w) = \frac{1}{2}\left(1 + \frac{1+w}{(1-w)^2 + 4wz}\right) \tag{36}$$

about $w = 0$.

Proof. Obtain from equation (33) the Schläfli integral representation,

$$y_{0k} = \frac{(1-z)k!}{2i} \oint_C \frac{(1-z')^{k-1} z'^k}{(z'-z)^{k+1}}\, dz' \tag{37}$$

where C is a closed path around z in the complex z plane leaving $z' = 1$ outside. Put it in the series. For sufficiently small w on C,

$$\left| \frac{(1-z')z'w}{z'-z} \right| < 1$$

we have

$$\sum_{k=0}^{\infty} \left[\frac{(1-z')z'w}{z'-z} \right]^k = \frac{1}{1-(1-z')z'w/(z'-z)}$$

Change summation and integration. Then

$$H(z, w) = (1-z) \frac{1}{2\pi i} \oint_C \frac{dz'}{(1-z')[z'-z-(1-z')z'w]} \tag{38}$$

According to the Rouché theorem, there is exactly one zero ζ of the denominator inside C,

$$\zeta(z, w) = \frac{1}{2w}\{-(1-w) + [(1-w)^2 + 4wz]^{1/2}\}$$

Then equation (36) is the residue of (38).

Corollary 5.1. From a comparison of equation (36) with the generating function for the Legendre polynomials P_k for $k \geq 1$ we obtain

$$y_{ok}(z) = (1/2)[P_k(1-2z) + P_{k+1}(1-2z)] \tag{39}$$

and

$$(-1)^k y_{0k}(z) + y_{0k}(\zeta) = P_k(z-\zeta) \tag{46}$$

where $\zeta = 1 - z$ [cf. equation (16)] and $y_{0k}(\zeta)$ is the conjugate polynomial with respect to the pair (13a)-(13b).

Corollary 5.2. The following recursion formulas can be obtained from equations (33) and (36):

$$y_{0k} = (1-2z)\frac{d}{dz} y_{0k} - \frac{1}{2}\frac{d}{dz} y_{0,k+1} - \frac{1}{2}\frac{d}{dz} y_{0,k-1}$$

$$y_{0,k+1} = \left[\frac{2(2k^2-1)}{(k+1)(2k-1)} - \frac{2(2k+1)}{k+1} z\right] y_{0k} - \frac{2(2k+1)(k-1)^2}{(2k-1)(2k-2)(k+1)} y_{0,k-1}$$

and, more generally for $m \geq 0$, $k \geq 1$,

$$y_{m,k+1} = (a + bz) y_{mk} + c y_{m,k-1}$$

where a, b, and c are constants depending on m and k. Further recursion formulas can be obtained from the known properties of the Jacobi polynomials.

6. PRODUCT DECOMPOSITION

This section shows how many components are obtained by a decomposition of a product of two y's.

Lemma 6.1. Let $q_l(z)$ be polynomials in z of degree l. Then the functions

$$Q_{ml}(z) =: z^{m/2}(1-z) q_l(z) \tag{41}$$

can be represented by

$$Q_{ml}(z) = \textstyle\sum_{\kappa=1}^{l+1} a_\kappa y_{m\kappa}(z) \tag{42}$$

and

$$Q_{ml}(z) = \sum_{\mu=0}^{l} b_\mu y_{m+2\mu, l+1-\mu}(z) \tag{43}$$

Now, consider that a product of any two y's is of the form (41),

$$y_{m_1k_1}\cdot y_{m_2k_2}=Q_{ml},\qquad m=m_1+m_2,\quad l=k_1+k_2-1$$

Example 6.1.:

$$\begin{aligned}y_{21}(z)y_{32}(z)&=a_1y_{51}(z)+a_2y_{52}(z)+a_3y_{53}(z)\\&=b_0y_{53}(z)+b_1y_{72}(z)+b_3y_{91}(z)\end{aligned}$$

The first row is for given m, and the second row is for given $\omega=m+2k$. From this it follows that the coefficients a_κ and b_μ can be obtained from Theorem 1.

Theorem 5. In the representation

$$y_{m_1k_1}(z)y_{m_2k_2}(z)=\sum_{\kappa=1}^{k_1+k_2}a_\kappa y_{m_1+m_2,\kappa}(z)\tag{44}$$

all coefficients vanish with

$$\kappa<\max\{k_1-m_2-k_2,\,k_2-m_1-k_1\}$$

Proof. Multiply by $\rho_1y_{m_1+m_2,\kappa'}(z)$ and integrate. Then, from Theorem 1,

$$a_{\kappa'}=N^{-1}\int_0^1\rho_1y_{m_1k_1}(z)y_{m_1+m_2,\kappa'}(z)y_{m_2k_2}(z)\,dz\tag{45}$$

where

$$N=\int_0^1\rho_1y^2_{m_1+m_2,\kappa'}(z)\,dz=\frac{\kappa'}{(\kappa'+m_1+m_2)(2\kappa'+m_1+m_2)}$$

Use equation (43),

$$y_{m_1,k_1}(z)y_{m_1+m_2,\kappa'}(z)=\sum_{\nu=1}^{m_1+k_1+\kappa'}b_\nu y_{m_2\nu}(z)$$

and put it into equation (45). Then, from Theorem 1 again, we see that $a_{\kappa'}=0$ for $m_1+k_1+\kappa'<k_2$, that is, $\kappa'<k_2-m_1-k_1$. Symmetrically, $a_{\kappa'}=0$ for $\kappa'<k_1-m_2-k_2$.

Example 6.2.:

$$\begin{aligned}y_{11}(z)y_{mk}(z)&=a_{k+1}y_{m+1,k+1}(z)+a_ky_{m+1,k}(z)\\&\quad+a_{k-1}y_{m+1,k-1}(z)+a_{k-2}y_{m+1,k-2}(z)\\21y_{11}y_{12}&=4y_{23}+6y_{22}\\5005y_{11}y_{16}&=792y_{27}+924y_{26}-420y_{25}-525y_{24}\end{aligned}$$

Corollary 6.1. The number of coefficients different from zero is always smaller than or equal to $\min\{\omega_1+1, \omega_2+1\}$.

It is noteworthy that, for even m_1, a similar theorem is valid also for a product of conjugated polynomials:

Theorem 6. Let m_1 be an even integer, $k_1, k_2 > 1$. In the representation

$$y_{m_1 k_1}(\zeta) y_{m_2 k_2}(z) = \sum_{\kappa=1}^{k_1+k_2-1+m_1/2} a_\kappa y_{2+m_2,\kappa}(z) \tag{46}$$

all the coefficients a_κ vanish with

$$\kappa < k_2 - k_1 - 1 - m_1/2$$

Proof. Equation (46) comes from equation (42) and the representation

$$y_{m_1 k_1}(\zeta) = z(1-z) q_{k_1-2+m_1/2}$$

Corollary 6.2. Only ω_1+1 terms are needed in the representation (46) maximally. For a development with $y(\zeta)$, a symmetrical representation is obtained.

7. ASYMPTOTIC REPRESENTATIONS

An asymptote can be obtained from a comparison of our differential equation (20) with the Bessel equation,

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0 \tag{47}$$

for $x \to 0$. Putting $s = \omega x$, we derive the following equation from equation (20) for $x \ll 1$:

$$s^2\, d^2\tilde{y}/ds^2 + s\, d\tilde{y}/ds + (s^2 - m^2)\tilde{y} = 0$$

Since $\tilde{y}_{0k}(0) = 1$ and $\tilde{y}_{mk}(0) = 0$ for $m \geq 1$,

$$\tilde{y}_{mk}(x) \approx \frac{(k+m-1)!}{(k-1)!(m/2+k)^m} J_m(\omega x) \qquad \text{for} \quad x^2 \ll 1 \tag{48}$$

where $\omega = m + 2k$ again and J_m is the Bessel function of first kind of order m.

Comparing the Legendre differential equation for $m = 0$,

$$(1-x^2) y'' - 2xy' + l(l+1) y = 0 \tag{49}$$

with our equation (20) for $m = 0$, rearranged in the form

$$(1-x^2)\tilde{y}'' + \frac{1-x^2}{x}\tilde{y}' + (2k)^2 \tilde{y} = 0 \tag{50}$$

we see that, for $k \gg 1$ and mean x values between 0 and 1, our $\tilde{y}_{0k}$ behaves like a solution of the Legendre equation (49) for $l = 2k$, because for $k \gg 1$ and $l \gg 1$ the second term of equations (49) and (50) can be neglected.

8. COMPLETENESS

Let

$$v =: (1-z)^{-1/2} y(z) \tag{51}$$

where $y(z)$ is a solution of our differential equation (17). Then, from (17), we have for v

$$[z(1-z)v']' + \left[-\frac{2-z}{4(1-z)} - \frac{m^2}{4z} + \frac{\omega^2}{4}\right] v = 0 \tag{52}$$

Theorem 7. The linear differential operator L_m, m a nonnegative integer,

$$L_m v(z) =: -[z(1-z)v']' + \left[\frac{2-z}{4(1-z)} + \frac{m^2}{4z}\right] v$$

with

$$D(L_m) = \{v \colon v(z) = z^{m/2}(1-z)^{1/2} p(z)\}$$

where p is a polynomial of z, is essentially self-adjoint in $L_2(0,1)$, the closure $\overline{L_m}$ is an operator with a pure point spectrum, the eigenvalues $\lambda_k = \omega_k^2/4 = (m+2k)^2/4$, $k = 1, 2, \ldots$, are single, and the corresponding eigenfunctions are

$$v_{mk} = z^{m/2}(1-z)^{1/2} F(m+k+1,\ -(k-1),\ m+1,\ z) \tag{53}$$

Proof. The last statements follow from

$$L_m v_{mk} = (m/2 + k)^2 v_{mk} \tag{54}$$

and the theorem follows with arguments given by Triebel (1972).

Theorem 7 is also valid for any real $m \geq 0$. Now, we consider the orthogonal system from Theorem 1(i) with $\rho_1 = (1-z)^{-1}$ for $m = 0, 1, 2, \ldots$. From Theorem 7 we have the following:

Theorem 8. The system

$$\{y_{mk}\}_{k=1}^{\infty}, \qquad m = 0, 1, 2, \ldots, \text{given}$$

is a complete orthogonal system on $[0, 1]$ with the weight $\rho_1 = (1-z)^{-1}$. Any function $h(z)$,

$$h(z) = g(z)(1-z)^{1/2} = g(z)\rho_1^{-1/2}, \qquad g(z) \in L_2(0,1)$$

can be represented as

$$h(z) = \sum_{k=1}^{\infty} a_k y_{mk}(z) \tag{55}$$

This theorem is also valid for any real $m \geq 0$.

The orthogonal systems from Theorem 1(ii) with given $\omega = m+2k$, $\omega \gtrsim 2$ integer, and $\rho_2 = z^{-1}(1-z)^{-1}$ are finite. The dimension of the corresponding functional spaces increases with increasing ω.

* * *

Ending the mathematical part of our paper, we wish to make a remark concerning the following question: Can a given function $f(\tau, \varphi_1, \varphi_2, \vartheta_3)$ on $S^1 \times S^3 \sim U(2)$ be represented as a series expansion over A_1^{mk} and A_2^{mk} according to equations (14a) and (14b)? This is not possible. Although we have really an eigenvalue problem with respect to the differential equation (17) and to a periodicity requirement on $U(2)$, it is not the eigenvalue problem suitable for that question. The latter should be of the form $M(U(2))(F) = \lambda_M F$ where M is some linear operator on $U(2)$ such as the Laplace-Beltrami operator or something like that. Clearly, the homogeneous equations (1) are not of such a form.

Our analysis shows that, apart from a periodicity requirement, the properties of the polynomials y_{mk} are rather independent of the exponential wave factors of equations (14),

$$\exp i(\omega\tau - m\varphi_j), \qquad j = 1, 2 \tag{56}$$

This means that the polynomials y_{mk} are to be considered as the proper hidden functions of our charge model.

9. DISCUSSION OF THE POSSIBILITY OF GIVING OUR MODEL A PHYSICAL SIGNIFICANCE

In this final section we propose a test of the physical significance of our charge model.

Historically, Einstein's suggestion (1909, paragraph 10; see also Dirac, 1963) that quantum physics be traced back to concepts of elementary charge has not been followed. Instead, the Planck constant $\hbar$ (uncertainty relation) was the starting point, and attempts were made to obtain quantized charges from it.

Now, in the spirit of Einstein, we will start from a large set of our $U(2)$ individuals [field configuration equations (14)] and, for the time being, most consistently, only from such a set, and will discuss the possibility of how to obtain the common space-time and a reasonable quantum theory from it. A rough sketch of a map on the standard Weinberg-Salam model for the electroweak interaction seems to be a fair choice for a first step.

The main problem is the handling of *nonlocal*, infinitely large individuals with the focus of *local* field theories ("particle points").

What follows is a short list of concepts and (unsolved) problems, which is given as a program for further work. A more detailed discussion is in preparation.

Charge. The $U(2)$ individuals (14) are considered to be a model for a dimensionless representation of naturally [by φ_i-periodicity in $U(2)$] quantized charges. They are characterized by two integers, m and k. The interpretation is as follows (Donth, 1984, 1986): The fields A_1 and A_2 have different charge signs; charge conjugation is

$$C: \quad x \leftrightarrow \xi, \quad \varphi_1 \leftrightarrow \varphi_2 \tag{57}$$

and therefore $A_1 \leftrightarrow A_2$, the net value of a charge is m, and k (a "private vacuum") can be used for indexing the lepton generations. Using the symbols

$$A_1^{mk} = \begin{pmatrix} m+ \\ k\mp \end{pmatrix}, \qquad A_2^{mk} = \begin{pmatrix} m- \\ k\pm \end{pmatrix} \tag{58}$$

(the sum of signs is $\omega = 2k + m$), we can describe the charged leptons $(e, \mu, \tau, \ldots)$ by compact waves

$$\begin{pmatrix} - \\ 0 \end{pmatrix}, \quad \begin{pmatrix} - \\ \pm \end{pmatrix}, \quad \begin{pmatrix} - \\ \pm\pm \end{pmatrix}, \ldots, \tag{59}$$

and the neutrinos by the corresponding compact oscillations $(m = 0)$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \pm \end{pmatrix}, \quad \begin{pmatrix} 0 \\ \pm\pm \end{pmatrix}, \ldots \tag{60}$$

These functions can also be used for quark flavors, and $\binom{-}{0}$ seems to be a good candidate for the baryon charge. It can be attached to the baryon as a whole, as the $U(2)$ individuals are assumed to be of infinite size. A full classification scheme with no need for broken quark charges (which could hardly be integrated into our model with φ_i periodicity) is given in Donth (1986). The existence of only two charge signs is a consequence of the transversality of our compact wave model [cf. equation (13c)].

Space-time. The common plane Minkowski space M_4 can only be constructed from plane "private" space elements, that is, from tangential spaces at the $U(2)$ individuals. The first problem is linking the two non-equivalent spinor representations of $SL(2, C)$ with the pairs (A_1, φ_2) and (φ_1, A_2) on the $U(2)$ configurations with no loss of the particle identities. The second problem is the reason for the metric (9) used in $U(2)$. Does it come from the presumed wave properties (Huygens principle)? (More radically: are there waves without any metrics?) Or does it come from some mutual local electrodynamic correspondence between the individual $U(2)$'s and the common M_4? If the light cone "is given" (e.g. stemming from bilinear spinor bases), then its invariance implies conformal symmetry of the common space (Weyl, 1923, p. 74).

The exponentials (56) are suggested to be the basis for the construction of a nontrivial connectedness of the common space. Using de Broglie's famous phase equality principle, one can try to construct an "undulatoric connection" from $\exp(i\omega\tau)$ that should lead to an affine connection in the classical limit. The second exponentials $\exp(im\varphi_j)$, $j = 1, 2$, containing the $SU(2)$ coordinates φ_1 and φ_2 and the net charge m, should be good candidates for constructing gauge connections.

The existence of individual space elements was the basis for a nonrelativistic proof of the spin-statistics theorem (Donth, 1970, 1977).

Quantum. The main concept of quantum theory is the "indivisibility" of its phenomena. Therefore, our model makes sense only when the $U(2)$ individuals are principally considered to be the quanta per se. Different quanta must be identifiable in M_4. This means that the "touch" between M_4 and the $U(2)$ individuals should be realized by a Lie algebra, which is, according to the Frobenius theorem, the integrability condition needed for their identification. As the definition of a commutator includes a certain neighborhood (i.e., the touch is more than a point), we should find here the basis for the uncertainty principle. As the interaction between particles is quantized, too, the gauge connections should also be linked to $U(2)$ individuals, e.g., two solutions (14) for a gauge boson.

Points in M_4. Local field theory means, in spite of quantization, that points can be defined in M_4. But a point can be constructed from extended space elements only when an infinite number of them can be used (cf. such constructions as Dedekind's cut, culminating point, or something similar). Therefore, the vacuum should be represented by "big" individuals (A_i^{mk} in the limit $k \to \infty$; or $k \to 10^{40}$, the rest being the gravitation?).

Let us define vacuum elements ("virtual neutrinos," "spinning oscillators") as $A_i^{0k'}$ functions with small k' *being members* (multiplicative elements) *of a series expansion* of the big A_i^{0k}, $k > k'$. In the example of the series (25c) we have the neutrinos ($k' = 1$)

$$\nu_z = \exp(2i\tau)z, \qquad \nu_\zeta = \exp(2i\tau')\zeta \tag{61}$$

for

$$A_1^{0k} = \cdots + (-1)^\kappa \binom{k}{\kappa}\binom{k-1}{\kappa} \nu_z^\kappa \nu_\zeta^{k-\kappa} + \ldots, \qquad \tau = \tau' \tag{62}$$

They are called virtual because neither ν_z^κ nor $\nu_\zeta^{k-\kappa}$ nor their product is a "true" individual, i.e., a solution of equations (10) in the sense of equations (14) for κ, $k-\kappa>1$.

Let us assume that a big individual is not stable in M_4 because the binomials in the series (25) (statistical weights of its vacuum elements) are steeply increasing with k [cardinality e^{ak}, $a=O(1)$]. Then there are enough vacuum elements for the construction of point spaces.

It seems to be an interesting question to find the "true" vacuum series expansion from a chemical equilibrium between vacuum elements of several generations. The presumed instability could also be the reason for a breakdown of the generation sequence (59) and (60) and the nonexistence of nonpolynomials as individuals.

Renormalization. The desired map on local fields in a $U(2)$-gauge fiber bundle over M_4 demands that (i) the $U(2)$ individuals (containing field configurations A_i^{mk}) could be reduced to group or generator representations (containing no field configurations) with respect to both M_4 and $U(2)^2$, and that (ii) the construction of a point in M_4 could be achieved by a condensation of the hidden polynomials y_{mk} to mass and charge values. Therefore, the concepts of point, mass value m_0, charge value e, and experiment (using the vacuum here) are tightly linked.

A strongly simplified model for calculating the charge value along these line is given in Donth (1986) using the statistics of Remark 2.1 and simplifying all the open geometric problems globally by a Hopf map from $U(2)$ to $S^2\subset R^3$. The result is a reduced coupling constant $g'\approx 0.09073$, and a vacuum symmetry breaking angle $\vartheta_3'\neq 0$ with $\sin^2\vartheta_3'\approx 0.2315$.

Weinberg–Salam Model. Three different realizations of the $U(2)$ individuals can be envisaged with respect to the quasiclassical Lagrangian:

1. Bundle section realizations ψ being group representations [spinors in M_4 and $U(2)$].
2. Bundle connection realizations A being generator representations (vectors).
3. An infinite stack of φ realizations for the vacuum elements.

Gauge invariance means that the A individuals can rotate freely as a whole with respect to the ψ and φ realizations. But, possibly, the mean relative rotation of the ψ's to the φ stack is fixed (breaking of the vacuum symmetry). Six angles would be needed to fix the corresponding, quasihidden tangential spaces. It seems to be an interesting question, whether there is a relation

[2]Idea. Hide the 1-forms on $U(2)$ by the use of Grassmannian numbers for spinor representations of the bundle sections: $(A_\mu x^\mu)^2=x_\mu x^\mu$.

between this or a similar hypothesis and the following six properties of the quasiclassical Lagrangian for the Weinberg-Salam model: (i) labeling of one isospin component, A^3 say, for combining with the hypercharge field by a suitable chiral formulation of the ψ's, (ii) the Weinberg angle ($\approx \vartheta_3'$?) realizing the correspondence to the photon, (iii) the 45°, rotation for obtaining $W^{\pm}$ from A^1 and A^2, and (iv–vi) the absorption of three Goldstone Higgs fields. As the fixed rotation has an orientation (R or L), we can possibly find here the reason for the chirality.

If there is no possibility for identifying single vacuum elements of the φ stack as individuals in M_4, then there is no sense in a search for φ quanta (Higgs particles).

Let us assume that there are individuals "confined" in complex boundaries [cf. the tables in Donth (1986)] or being additionally virtual in the vacuum. If they can do without their "own" weakly attached exponentials (56), then, possibly, they can do with collective "common" exponentials now having three more or less equivalent compact coordinates $\tau, \varphi_1, \varphi_2$ ("colors") for living with a limited identity defined only by their polynomials y_{mk} (instead of A_i^{mk}).

The existence of massless photons in M_4 is a requirement of the correspondence principle between our highly nonlocal $U(2)$ wave field configurations and free Hertz waves in M_4.

In conclusion, we have some hope of finding a reasonable quantum theory starting from highly nonlocal charge models because we have seen such rich and complex construction possibilities, even in our simple model, without leaving concepts in the actual order of magnitude (for instance no additional length is introduced or referred to).

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REFERENCES

Arnol'd, V. I. (1984). *Obyknovennye differencial'nye uravneniya*, 3rd ed., Nauka, Moscow (in Russian).

Barut, A. O., and Raczka, P. (1977). *Theory of Group Representations and Applications*, Scientific Publishers, Warsaw).

Bateman, H., and Erdélyi, A. (1953). *Higher Transcendental Functions*, Vol. 2, McGraw-Hill, New York.

Biess, G., Donth, E., and Jänicke, W. (1986). *Wissenschaftliche Zeitschrift der Technischen Hochschule Leuna-Merseburg*, in press.

Dirac, P. (1963). *Scientific American*, **1963** (May).

Donth, E. (1970). *Physics Letters*, **32A**, 209.

Donth, E. (1977). *Wissenschaftliche Zeitschrift der Technischen Hochschule Leuna-Merseburg*, **19**, 602.

Donth, E. (1984). *Experimentelle Technik der Physik*, **32**, 213.

Donth, E. (1986). *Wissenschaftliche Zeitschrift der Technischen Hochschule Leuna-Merseburg*, in press.

Einstein, A. (1909). *Physikalische Zeitschrift*, **10**, 185.

Szegö, G. (1959). *Orthogonal Polynomials*, American Mathematical Society, New York.

Triebel, H. (1972). *Höhere Analysis*, Deutscher Verlag der Wissenschaften, Berlin.

Weyl, H. (1923). *Mathematische Analyse des Raumproblems*, Springer, Berlin.